Entanglement and statistical independence for mixed quantum states

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We show that three conditions associated with 'entanglement' – non-locality, non-factorisability and statistical dependence – are equivalent for pure states, and that non-factorisability and statistical dependence are equivalent for mixed states. Discussion then reinforces the generally held view that the key condition for mixed states is non-separability.

1 Introduction.

The term 'entangled', originally referring to the situation where a pure state of a joint system is not a tensor product ([5], p 184), is sometimes used more to characterize situations showing quantum non-locality. The classic example of entanglement, in Bohm's simplified version of the Einstein-Rosen-Podolsky phenomenon ([3], p 198 ff.), is the singlet state of a pair of spin 1/2 identical particles

$$|u\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

which is an 'entangled state' relative to the decomposition of the Hilbert space into the tensor product of two 1-particle systems, in the sense that $|u\rangle$ is not a tensor product. The converse of entangled is *separable*, defined as being a tensor product for pure states, or [4] a convex sum of tensor products for mixed states.

To consider a wider range of conditions related to separability, let us take a space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that is the tensor product of the spaces for two systems A and B, and let $|v\rangle \in \mathcal{H}$. Then three possible conditions might be used to capture the idea that $|v\rangle$ does not link together the two systems. (In the following $\langle Q\rangle_v$ denotes the expectation of the observable Q in the state $|v\rangle$ – i.e. $\langle Q\rangle_v = \langle v|\hat{Q}|v\rangle$.)

1. (Factorisability) $|v\rangle = |p\rangle|q\rangle$ for some $|p\rangle \in \mathcal{H}_A$ and $|q\rangle \in \mathcal{H}_B$

2. (The generalized Bell Inequality [2]) For all propositions P, P' on A

and Q, Q' on B, if we define $R = 2P - 1$, $S = 2Q - 1$, $R' = 2P' - 1$, $S' = 2Q' - 1$ then the inequality

$$|\langle RS \rangle_v + \langle RS' \rangle_v + \langle R'S \rangle_v - \langle R'S' \rangle_v| \leq 2$$

holds.

3. (Statistical independence) For all propositions P, Q on A, B respectively

$$\langle PQ \rangle_v = \langle P \rangle_v \langle Q \rangle_v. \tag{1}$$

The contexts of these conditions tend to be different: factorisabilty is an essentially mathematical condition, Bell's inequality concerns locality in general systems (not necessarily quantum systems), and independence refers to the purely statistical level without reference to underlying mechanisms. Nonetheless, a combination of known results and simple arguments shows that:

Proposition 1. Conditions 1 - 3 above become equivalent when applied to pure states $|v\rangle$.

Proof

$1 \Rightarrow 3$. Suppose $|v\rangle = |p\rangle|q\rangle$ as in 1 and let P, Q be propositions as in 3.

Then

$$\begin{aligned}\langle PQ\rangle_v &= \langle v|(\hat{P}\otimes 1)(1\otimes\hat{Q})|v\rangle = \langle p|\hat{P}|p\rangle\langle q|\hat{Q}|q\rangle \\ &= \langle P\rangle_v\langle Q\rangle_v\end{aligned}$$

$3\Rightarrow 2$. Let v be such that 3 holds, and let R, R', S, S' be as in 2. Then, using $|\langle R\rangle_v|\leq 1$ and similarly for R', S, S', we have

$$\begin{aligned}|\langle RS\rangle_v &+ \langle RS'\rangle_v + \langle R'S\rangle_v - \langle R'S'\rangle_v| = |\langle R\rangle_v(\langle S\rangle_v + \langle S'\rangle_v) + \langle R'\rangle_v(\langle S\rangle_v - \langle S'\rangle_v)| \\ &\leq |(\langle S\rangle_v + \langle S'\rangle_v)| + |(\langle S\rangle_v - \langle S'\rangle_v)| \\ &\leq \max((\langle S\rangle_v + \langle S'\rangle_v) + (\langle S\rangle_v - \langle S'\rangle_v), (\langle S\rangle_v + \langle S'\rangle_v) - (\langle S\rangle_v - \langle S'\rangle_v), \\ &\quad -(\langle S\rangle_v + \langle S'\rangle_v) + (\langle S\rangle_v - \langle S'\rangle_v), -(\langle S\rangle_v + \langle S'\rangle_v) - (\langle S\rangle_v - \langle S'\rangle_v)) \\ &= \max(2\langle S\rangle_v, 2\langle S'\rangle_v, -2\langle S'\rangle_v, -2\langle S\rangle_v) \leq 2\end{aligned}$$

$2\Rightarrow 1$. This is a result of [1], or see Home, [3] page 205 ff.

QED

While this is quite interesting, the main aim of this paper is to explore the trickier situation that arises when $|v\rangle$ is replaced by α, a *mixed* state of the joint system composed of A and B, i.e. a unit trace Hermitian operator on $\mathcal{H}_A\otimes\mathcal{H}_B$. Then properties 2 and 3 can be repeated unchanged provided that we now define, as usual, $\langle Q\rangle_\alpha = \mathrm{Tr}\,\hat{Q}\alpha$.

Factorisability and independence take the forms

$1'$. (Factorisability) $\alpha = \sigma \otimes \rho$ for mixed states σ and ρ on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

$3'$. (Statistical independence) For all propositions P, Q on A, B respectively

$$\langle PQ \rangle_\alpha = \langle P \rangle_\alpha \langle Q \rangle_\alpha. \tag{$*$}$$

It is now a key aspect of the arguments around Bell's inequality that, while $1'$ implies 2 (the proof is elementary), the converse does not hold. This is because a mixture of product states will satisfy Bell's inequality, but will in general not itself be a product state as defined above. It is, of course, for this reason that "separability" is defined as being a convex sum of product states, rather than being simply a product state. We can, however, still prove the other part of the equivalences in Proposition 1, our main result being:

Proposition 2. Conditions $1'$ and $3'$ above are equivalent for all mixed states α.

Proof

$1' \Rightarrow 3'$. Suppose $\alpha = \sigma \otimes \rho$ and let P, Q be propositions as in $3'$. Then

$$\begin{aligned} \langle PQ \rangle_\alpha &= \mathrm{Tr}\,[(\hat{P} \otimes \hat{Q})(\sigma \otimes \rho)] = \mathrm{Tr}\,\hat{P}\sigma\ \mathrm{Tr}\,\hat{Q}\rho \\ &= \langle P \rangle_\alpha \langle Q \rangle_\alpha \end{aligned}$$

$3' \Rightarrow 1'$. Let α be such that $3'$ holds.

> *Notation* For notational convenience we drop the Dirac notation and choose orthonormal bases $(e^i)_{i=1,2,\ldots}$ and $(f^i)_{i=1,2,\ldots}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Vectors are then expressed in terms of components, using the Einstein summation convention throughout, as $v = v_i e^i$ (and similarly for B). Also if $w \mapsto \bar{w}$ denotes the anti-isomorphism between a Hilbert space and its dual, then the dual bases can be written as $(\bar{e}_i)_{i=1,2,\ldots}$ and $(\bar{f}_i)_{i=1,2,\ldots}$ with $\bar{v}^i = \overline{v_i}$. Here we use the usual convention that raising indices denotes quantities transforming under the transposed inverse of the representation holding for lowered indices, which for unitary transformations is the same as the conjugate representation.

Then we can write

$$\alpha = \alpha_{ip}{}^{jq} e^i \otimes f^p \otimes \bar{e}_j \otimes \bar{e}_q.$$

Choose for P and Q projections on the 1-dimensional subspaces spanned by unit vectors $a = a_i e^i$ and $b = b_j f^j$. Then $(*)$ in 3 is

$$\alpha_{jq}{}^{ip}\bar{a}^j\bar{b}^q a_i b_p - \alpha_{jp}{}^{ip}\bar{a}^j a_i \alpha_{lq}{}^{lr}\bar{b}^q b_r = 0. \tag{2}$$

Write (2) as $F(x_i, y_i, w_j, z_j) = 0$, where $a_i = x_i + \mathrm{i}y_i$, $b_j = w_j + \mathrm{i}z_j$. Then this expression is valid for all x_i, y_i, w_j, z_j satisfying $S_1(x_i, y_i, w_j, z_j) \equiv a_i\bar{a}^i - 1 = 0$, and $S_2(x_i, y_i, w_j, z_j) \equiv b_j\bar{b}^j - 1 = 0$. Thus, with F, S_1 and S_2 as real functions on a real topological vector space regarded as manifold, we see that

$$dF = \lambda\, dS_1 + \mu\, dS_2 \tag{3}$$

for real numbers λ and μ depending on a and b.

Let $\partial/\partial\bar{a}^i$ be the operator $\partial/\partial x_i + \mathrm{i}\partial/\partial y_i$, which acts formally by "partially differentating with respect to $\bar{a}^i$ keeping a_i constant", and similarly for b. Then by taking combinations of the components of (3) we have

$$\frac{\partial F}{\partial \bar{a}^m} = \lambda \frac{\partial S_1}{\partial \bar{a}^m} + \mu \frac{\partial S_2}{\partial \bar{a}^m}.$$

Evaluating this:

$$\alpha_{mq}{}^{ip}\bar{b}^q a_i b_p - \alpha_{mp}{}^{ip} a_i \alpha_{lq}{}^{lr}\bar{b}^q b_r = \lambda(a,b) a_m. \tag{4}$$

where the argument (a, b) denotes unspecified dependence on the real and imaginary parts of the components of the vectors concerned.

We now repeat this argument, applying the operator $\partial/\partial a_n = \partial/\partial x_n - \mathrm{i}\partial/\partial y_n$ to (4), and introducing a further (m-dependent) Lagrange multiplier μ_m, giving

$$\alpha_{mq}{}^{np}\bar{b}^q b_p - \alpha_{mp}{}^{np}\alpha_{lq}{}^{lr}\bar{b}^q b_r = \frac{\partial\lambda(a,b)}{\partial a_n} a_m + \mu_m \bar{a}^n \tag{5}$$

We now fix b for the time being, suppressing reference to this argument explicitly, and abbreviate (5) as

$$S_m{}^n = h^n(a) a_m + k_m(a)\bar{a}^n. \tag{5'}$$

Choose a fixed dual vector w^m and choose a so that $a_m w^m = 0$. Then we note that from (5)

$$w^m S_m{}^n = \zeta(a)\bar{a}^n \qquad \text{for all } a \text{ with } a_m w^m = 0. \tag{6}$$

Since the left hand side is independent of a and the right hand side is continuous in a we must have $\zeta = 0$ and hence, since w was arbitrary, $S = 0$. That is

$$(\alpha_{mq}{}^{np} - \sigma_m{}^n \rho_q{}^p)\bar{b}^q b_p = 0 \tag{7}$$

where

$$\sigma_m{}^n := \alpha_{mr}{}^{nr}, \qquad \rho_q{}^p := \alpha_{lq}{}^{lp}.$$

Now, considering (7), we note that, whereas the *real* linear space generated by the quantities $\bar{b}^q b_p$ consists of all bounded Hermitian operators, the *complex* linear space generated by these is dense in the space of *all* bounded operators, Hermitian or not. (This is fundamentally different from the real case of equations of the form (7).) As a result, (7) implies that

$$\alpha_{mq}{}^{np} - \sigma_m{}^n \rho_q{}^p = 0$$

as required.

QED

2 Discussion.

Does the forgoing shed any light on this group of concepts as applied to both pure and mixed states? Physical interest attaches primarily to the fact of a correlation coming from non-locality rather than from an antecedent common cause. This is not captured by statistical dependence (violation of $3'$) because, as indicated by this result, this can arise from a mixture of un-entangled states (a separable state), which could result from an antecedent common cause. Moreover, it has been known for some time that Bell's in-equality is not sufficiently sensitive to detect all non-separable states, because of the example of Werner's state [6] which is non-separable but satisfies Bell's inequality.

The result here thus places statistical dependence lower down within a hierarchy of related conditions, while supporting the position of non-separability as the main contender for a mathematical characterisation of non-locality.

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3 References

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